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On Lambert series and continued fractions

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Abstract: In this paper, we have established certain results involving Lambert series and continued fractions.

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1. Introduction, Notations and Definitions

The q-shifted factorial is defined by,

$$(a, q)_n = \begin{cases} 0 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\ldots,(1-aq^{n-1}) & \text{if } n \geq 1. \end{cases}$$

Also,

$$(a; q)_{-n} = \frac{q^{n(n+1)/2}}{(-a)^n (q/a; q)_n}.$$

The generalized basic hypergeometric series is given by

$${}_{r+1}\Phi_r \left[\begin{array}{l} a_1, a_2, ..., a_{r+1}; q; z \\ b_1, b_2, ..., b_r \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_{r+1}; q)_n z^n}{(q, b_1, b_2, ..., b_r; q)_n},$$

where $(a_1, a_2, ..., a_{r+1}; q)_n = (a_1; q)_n (a_2; q)_n ... (a_{r+1}; q)_n$ and max.$(|z|, |q|) < 1$ for the convergence of the series.

The generalised bilateral basic hypergeometric series is defined as,

$${}_r\Psi_r \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_r \end{array} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(b_1, b_2, ..., b_r; q)_n},$$

where $\left| \frac{b_1, b_2, ..., b_r}{a_1, a_2, ..., a_r} \right| < |z| < 1$ for the convergence of the series.
An expression of the form,

$$b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3 + ...}$$

is called infinite continued fraction.
The series of the form,

$$\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n} \tag{1.1}$$

is called Lambert series. If $\sum_{n=1}^{\infty} a_n$ converges, then the Lambert series (1.1) converges for all values of z except for $z = \pm 1$; otherwise it converges for those values of z for which the series $\sum_{n=1}^{\infty} a_n z^n$ converges.

There are a number of fascinating results on continued fractions in the notebooks of Srinivasa Ramanujan, specially in the second and 'Lost' notebooks. Many mathematicians such as B. Gorden [7], L. Carlitz [4], Hirschhorn [9], K.G. Ramanathan [11], G.E. Andrews and B.C. Berndt [1], S. Bhargava and C. Adiga [2], D.P. Gupta and D. Masson [8], R.Y. Denis [5,6], S.N. Singh [12], N. Bhagirathi [3] and many others have proved and also generalized the results of Ramanujan on continued fractions.

In this paper, an attempt has been made to establish some interesting and new results involving Lambert series and continued fractions

2. Main Results

In this section we shall establish our main results.

(i)

$$\sum_{n=0}^{\infty} \frac{\lambda^n q^{in}}{1-\mu q^{kn+j}} = \sum_{n=0}^{\infty} \frac{q^{kn^2+(i+j)n}(\lambda\mu)^n(1-\lambda\mu q^{2kn+i+j})}{(1-\mu q^{kn+j})(1-\lambda q^{kn+i})}$$

$$= \frac{1}{(1-\mu q^j)-} \frac{\lambda q^i(1-\mu q^j)^2}{(1-\mu q^{k+j})-} \frac{\lambda\mu q^{i+j}(1-q^k)^2}{(1-\mu q^{2k+j})-} \frac{\lambda q^{k+i}(1-\mu q^{k+j})^2}{(1-\mu q^{3k+j})-}$$

$$\frac{\lambda\mu q^{k+i+j}(1-q^{2k})^2}{(1-\mu q^{4k+j})-} \frac{\lambda q^{2k+i}(1-\mu q^{2k+j})^2}{(1-\mu q^{5k+j})-} \frac{\lambda\mu q^{2k+i+j}(1-q^{3k})^2}{(1-\mu q^{6k+j}) - \ldots} \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{\lambda^n q^{in}}{1-\mu q^{kn+j}} = \sum_{n=0}^{\infty} \frac{q^{kn^2+(i+j)n}(\lambda\mu)^n(1-\lambda\mu q^{2kn+i+j})}{(1-\mu q^{kn+j})(1-\lambda q^{kn+i})}$$

$$-\sum_{n=0}^{\infty} \frac{q^{kn^2+(2k-i-j)n+k-i-j}(1-q^{2kn+2k-i-j}/\lambda\mu)}{(1-q^{kn+k-i}/\lambda)(1-q^{kn+k-j}/\mu)(\lambda\mu)^{n+1}}$$

$$= \frac{(q^k;q^k)^2(\lambda\mu q^{i+j}, q^{k-i-j}/\lambda\mu; q^k)_\infty}{(\mu q^j, \lambda q^i, q^{k-j}/\mu, q^{k-i}/\lambda; q^k)_\infty}. \tag{2.2}$$

Proof of (2.1)

In order to prove (2.1) let us consider the ratio

$$\frac{\sum_{n=0}^{\infty}\frac{(aq,b;q)_n}{(q,c;q)_n}z^n}{\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,c;q)_n}z^n}=\cfrac{1}{1-\cfrac{\sum_{n=0}^{\infty}\frac{(b;q)_n z^n}{(q;q)_n(c;q)_n}\{(aq;q)_n-(a;q)_n\}}{\sum_{n=0}^{\infty}\frac{(aq,b;q)_n}{(q,c;q)_n}z^n}}$$

$$=\cfrac{1}{1-\cfrac{az(1-b)/(1-c)}{1-\cfrac{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}z^n\left\{\frac{(bq;q)_n}{(cq;q)_n}-\frac{(b;q)_n}{(c;q)_n}\right\}}{\sum_{n=0}^{\infty}\frac{(aq,bq;q)_n}{(q,cq;q)_n}z^n}}}$$

$$=\cfrac{1}{1-\cfrac{az(1-b)/(1-c)}{1-\cfrac{z(b-c)(1-aq)/(1-z)(1-cq)}{1-\cfrac{\sum_{n=0}^{\infty}\frac{(bq;q)_n}{(q;q)_n}z^n\left\{\frac{(aq^2;q)_n}{(cq^2;q)_n}-\frac{(aq;q)_n}{(cq;q)_n}\right\}}{\sum_{n=0}^{\infty}\frac{(aq^2,bq;q)_n}{(q,cq^2;q)_n}z^n}}}}$$

$$\cfrac{1}{1-\cfrac{az(1-b)/(1-c)}{1-\cfrac{z(b-c)(1-aq)/(1-c)(1-cq)}{1-\cfrac{zq(a-c)(1-bq)/(1-cq)(1-cq^2)}{1-\cfrac{\sum_{n=0}^{\infty}\frac{(aq^2;q)_n}{(q;q)_n}z^n\left\{\frac{(bq^2;q)_n}{(cq^3;q)_n}-\frac{(bq;q)_n}{(cq^2;q)_n}\right\}z^n}{\sum_{n=0}^{\infty}\frac{(aq^2,bq^2;q)_n}{(q,cq^3;q)_n}z^n}}}}}$$

Iterating this process we have

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(aq,b;q)_n}{(q,c;q)_n}z^n}{\displaystyle\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(q,c;q)_n}z^n}=\frac{1}{1-}\,\frac{az(1-b)/(1-c)}{1-}\,\frac{z(b-c)(1-aq)/(1-c)(1-cq)}{1-}$$

$$\frac{zq(a-c)(1-bq)/(1-cq)(1-cq^2)}{1-}\,\frac{zq(b-cq)(1-aq^2)/(1-cq^2)(1-cq^3)}{1-}$$

$$\frac{zq^2(a-cq)(1-bq^2)/(1-cq^3)(1-cq^4)}{1-\ldots} \tag{2.3}$$

Taking a=1, making use of Rogers-Fine identity [1; (9.1.1) p. 223] and then applying [10; (2.3.14) p. 33] in (2.3) we find,

$$\sum_{n=0}^{\infty}\frac{(b;q)_n z^n}{(c;q)_n}=\sum_{n=0}^{\infty}\frac{(b;q)_n(bzq/c;q)_n c^n z^n q^{n^2-n}(1-bzq^{2n})}{(c;q)_n(z;q)_{n+1}}$$

$$=\frac{1}{1-}\,\frac{z(1-q)}{(1-c)-}\,\frac{z(b-c)(1-q)}{(1-cq)-}\,\frac{zq(1-c)(1-bq)}{(1-cq^2)-}\,\frac{zq(b-cq)(1-q^2)}{(1-cq^3)-}$$

$$\frac{zq^2(1-cq)(1-bq^2)}{(1-cq^4)-}\,\frac{zq^2(b-cq^2)(1-q^3)}{(1-cq^5)-\ldots} \tag{2.4}$$

Again, taking c=bq in (2.4) we get after some simplification,

$$\sum_{n=0}^{\infty}\frac{z^n}{1-bq^n}=\sum_{n=0}^{\infty}\frac{q^{n^2}(bz)^n(1-bzq^{2n})}{(1-bq^n)(1-zq^n)}=\frac{1}{(1-b)-}\,\frac{z(1-b)^2}{(1-bq)-}$$

$$\frac{bz(1-q)^2}{(1-bq^2)-}\,\frac{zq(1-bq)^2}{(1-bq^3)-}\,\frac{bzq(1-q^2)^2}{(1-bq^4)-}\,\frac{zq^2(1-bq^2)^2}{(1-bq^5)-}\,\frac{bzq^2(1-q^3)^2}{(1-bq^6)-\ldots} \tag{2.5}$$

Now, replacing q by q^k z by λq^i and b by μq^j in (2.5) we get (2.1).

Proof of (2.2)

In order to prove (2.2) let us consider the Ramanujan's celebrated summation formula, viz.,

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(q,b/a,az,q/az;q)_\infty}{(q,q/a,z,b/az;q)_\infty}, \tag{2.6}$$

Slater, L.J. [13; App. IV (IV. 12)]

which can be written as,

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n z^n}{(b;q)_n}=\sum_{n=0}^{\infty}\frac{(a;q)_n z^n}{(b;q)_n}+\frac{(1-q/b)}{(1-q/a)}\left(\frac{b}{az}\right)\sum_{n=0}^{\infty}\frac{(q^2/b;q)_n}{(q^2/a;q)_n}\left(\frac{b}{az}\right)^n$$

$$=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}. \tag{2.7}$$

Using Rogers-Fine identity [1; (9.1.1) p. 223] in (2.7) we have,

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\sum_{n=0}^{\infty}\frac{(a;q)_n(azq/b;q)_n b^n z^n q^{n^2-n}(1-azq^{2n})}{(b;q)_n(z;q)_{n+1}}$$

$$+\frac{(1-q/b)}{(1-q/a)}\left(\frac{b}{az}\right)\sum_{n=0}^{\infty}\frac{(q^2/b;q)_n(q/z;q)_n\left(\frac{bq^2}{a^2z}\right)^n q^{n^2-n}\left(1-\frac{q^{2n+2}}{az}\right)}{(q^2/a;q)_n(b/az;q)_{n+1}}$$

$$=\frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}. \tag{2.8}$$

Putting b=aq in (2.8) we get

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-aq^n}=\sum_{n=0}^{\infty}\frac{q^{n^2}(az)^n(1-azq^{2n})}{(1-aq^n)(1-zq^n)}-\frac{q}{az}\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(1-q^{2n+2}/az)(1/az)^n}{(1-q^{n+1}/a)(1-q^{n+1}/z)}$$

$$=\frac{(q;q)_\infty^2(az,q/az;q)_\infty}{(a,q/a,z,q/z;q)_\infty}. \tag{2.9}$$

Now replacing q by q^k, z by λq^i and a by μq^j in (2.9) we get (2.2)

3. Special Cases

In this section we shall deduce certain interesting special cases of (2.1) and (2.2).

(i) Taking $\lambda=\mu$ and $j=i$ in (2.1) we get

$$\sum_{n=0}^{\infty}\frac{\lambda^n q^{in}}{(1-\lambda q^{kn+i})}=\sum_{n=0}^{\infty}\frac{q^{kn^2+2in}\lambda^{2n}(1+\lambda q^{kn+i})}{(1-\lambda q^{kn+i})}$$

$$=\frac{1}{(1-\lambda q^i)-}\,\frac{\lambda q^i(1-\lambda q^i)^2}{(1-\lambda q^{k+i})-}\,\frac{\lambda^2 q^{2i}(1-q^k)^2}{(1-\lambda q^{2k+i})-}$$

$$\frac{\lambda q^{k+i}(1-\lambda q^{k+i})^2}{(1-\lambda q^{3k+i})-}\frac{\lambda^2 q^{k+2i}(1-q^{2k})^2}{(1-\lambda q^{4k+i})-}\frac{\lambda q^{2k+i}(1-\lambda q^{2k+i})^2}{(1-\lambda q^{5k+i})-}\frac{\lambda^2 q^{2k+2i}(1-q^{3k})^2}{(1-\lambda q^{6k+i})-\ldots} \tag{3.1}$$

(ii) Taking $\lambda=-1,\mu=1,i=j=2,k=4$ in (2.1) and using [1; entry (18.2.4) p. 397] we get

$$\Psi^2(q^4)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n}}{(1-q^{4n+2})}=\sum_{n=0}^{\infty}\frac{q^{4n^2+4n}(-1)^n(1+q^{8n+4})}{(1-q^{8n+4})}$$

$$=\frac{1}{(1-q^2)+}\frac{q^2(1-q^2)^2}{(1-q^6)+}\frac{q^4(1-q^4)^2}{(1-q^{10})+}\frac{q^6(1-q^6)^2}{(1-q^{14})+}\frac{q^8(1-q^8)^2}{(1-q^{18})+\ldots} \tag{3.2}$$

(iii) Again, taking $\lambda=1,\mu=-1,i=j=1,k=2$ in (2.1) and using [1; entry (18.2.4) p. 397] we find,

$$\Psi^2(q^2)=\sum_{n=0}^{\infty}\frac{q^n}{(1+q^{2n+1})}=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}(-1)^n(1+q^{4n+2})}{(1-q^{4n+2})}$$

$$=\frac{1}{(1+q)-}\frac{q(1+q)^2}{(1+q^3)+}\frac{q^2(1-q^2)^2}{(1+q^5)-}\frac{q^3(1+q^3)^2}{(1+q^7)+}\frac{q^4(1-q^4)^2}{(1+q^9)-}\frac{q^5(1+q^5)^2}{(1+q^{11})+\ldots} \tag{3.3}$$

Now, comparing (3.2) and (3.3) we get the fundamental property of Lambert series, viz.,

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n}}{(1-q^{4n+2})}=\sum_{n=0}^{\infty}\frac{q^{2n}}{(1+q^{4n+2})} \tag{3.4}$$

(iv) Taking $k=4,i=j=1,\lambda=\mu=1$ in (2.2) and then using [1; (6.2.25) p. 151] we find,

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{(1-q^{4n+1})}=\sum_{n=0}^{\infty}\frac{q^{4n^2+2n}(1+q^{4n+1})}{(1-q^{4n+1})}-\sum_{n=0}^{\infty}\frac{q^{4n^2+6n+2}(1+q^{4n+3})}{(1-q^{4n+3})}$$

$$=\left\{\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\right\}=\Psi^2(q)$$

$$=\left\{\frac{1}{1-}\frac{q}{1+}\frac{q-q^2}{1-}\frac{q^3}{1+}\frac{q^2-q^4}{1-}\frac{q^5}{1+\ldots}\right\}^2 \tag{3.5}$$

Comparing (3.3) and (3.5) we obtain

$$\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{(1-q^{8n+2})}=\sum_{n=0}^{\infty}\frac{q^n}{1+q^{2n+1}} \tag{3.6}$$

Also, comparing (3.5) and (3.2) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{(1-q^{16n+4})} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}}{1-q^{4n+2}} \tag{3.7}$$

(v) Taking $k=4, i=2, j=1, \lambda=\mu=1$ in (2.2) we get,

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{4n+1})} = \sum_{n=0}^{\infty} \frac{q^{4n^2+3n}(1-q^{8n+3})}{(1-q^{4n+1})(1-q^{4n+2})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+5n+1}(1-q^{8n+5})}{(1-q^{4n+2})(1-q^{4n+3})}$$

$$= \frac{(q^2;q^2)_\infty^2}{(q^2;q^4)_\infty^4} \tag{3.8}$$

Dividing (3.8) by (3.5) and using [1; (6.2.1) p. 150] we get

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{4n+1})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{n}}{(1-q^{4n+1})}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{4n^2+3n}(1-q^{8n+3})}{(1-q^{4n+1})(1-q^{4n+2})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+5n+1}(1-q^{8n+5})}{(1-q^{4n+2})(1-q^{4n+3})}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{4n^2+2n}(1+q^{4n+1})}{(1-q^{4n+1})} - \sum_{n=0}^{\infty} \frac{q^{4n^2+6n+2}(1+q^{4n+3})}{(1-q^{4n+3})}}$$

$$= \left\{ \frac{(q;q^2)_\infty}{(q^2;q^4)_\infty} \right\}^2 = \left\{ \frac{1}{1+}\, \frac{q}{1+}\, \frac{q+q^2}{1+}\, \frac{q^3}{1+}\, \frac{q^2+q^4}{1+\ldots} \right\}^2 \tag{3.9}$$

(vi) Choosing $k=5, i=j=1, \mu=\lambda=1$ in (2.2) we get,

$$\sum_{n=-\infty}^{\infty} \frac{q^{n}}{(1-q^{5n+1})} = \sum_{n=0}^{\infty} \frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}$$

$$= (q^5;q^5)_\infty^2 \frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty^2}. \tag{3.10}$$

Again, choosing $k=5, \lambda=\mu=1, i=j=2$ in (2.2) we obtain,

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{5n+2})} = \sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}$$

$$= (q^5;q^5)_\infty^2 \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty^2}. \tag{3.11}$$

Dividing (3.10) by (3.11) and then using [1; corollary (6.2.6) p. 153] we find,

$$\frac{\sum_{n=-\infty}^{\infty}\frac{q^n}{(1-q^{5n+1})}}{\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{(1-q^{5n+2})}}=\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}$$

$$=\left\{\frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}\right\}^3=\left\{\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+...}\right\}^3, \tag{3.12}$$

Andrews and Berndt[1; (4.4.2) p. 117]

which is the celebrated result of Ramanujan.
(vii) Choosing $\lambda=\mu=1, k=5, i=1, j=3$ in (2.2) we find,

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{(1-q^{5n+3})}=\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+3})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}$$

$$=\frac{(q^5;q^5)_\infty^2(q,q^4;q^5)_\infty}{(q,q^2,q^3,q^4;q^5)_\infty}=\frac{(q^5;q^5)_\infty^2}{(q^2,q^3;q^5)_\infty}. \tag{3.13}$$

Dividing (3.10) by (3.13) we get

$$\frac{\sum_{n=-\infty}^{\infty}\frac{q^n}{(1-q^{5n+1})}}{\sum_{n=-\infty}^{\infty}\frac{q^{n}}{(1-q^{5n+3})}}=\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+3})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}}$$

$$=\left\{\frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}\right\}^2=\left\{\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+...}\right\}^2. \tag{3.14}$$

Again, dividing (3.11) by (3.13) we obtain,

$$\frac{\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{(1-q^{5n+2})}}{\sum_{n=0}^{\infty}\frac{q^{n}}{(1-q^{5n+3})}}=\frac{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}{\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+3})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}}$$

$$= \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty} = 1 + \frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+}\frac{q^5}{1+...}. \tag{3.15}$$

(viii) Choosing $\lambda = \mu = 1, i = 1, j = 2$ and k=5 in (2.2) we find

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{5n+2})} = \sum_{n=0}^{\infty} \frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}$$

$$= \frac{(q^5;q^5)_\infty^2 (q^2,q^3;q^5)_\infty}{(q,q^2,q^3,q^4;q^5)_\infty} = \frac{(q^5;q^5)_\infty^2}{(q,q^4;q^5)_\infty}. \tag{3.16}$$

Dividing (3.16) by (3.13) we get,

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{5n+2})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{5n+3})}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+3})(1-q^{5n+1})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}}$$

$$= \frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+...}. \tag{3.17}$$

Dividing (3.10) by (3.16) we get,

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{5n+1})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{5n+2})}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}$$

$$= \frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+...}. \tag{3.18}$$

Again, dividing (3.11) by (3.16) we get,

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{(1-q^{5n+2})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{(1-q^{5n+2})}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})} - \sum_{n=0}^{\infty} \frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}$$

$$= \left\{\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}\right\}^2 = \left\{1 + \frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+...}\right\}^2. \tag{3.19}$$

(ix) Taking $\lambda=\mu=1, i=j=1, k=6$ in (2.2) we have

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{(1-q^{6n+1})}=\sum_{n=0}^{\infty}\frac{q^{6n^2+2n}(1+q^{6n+1})}{(1-q^{6n+1})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+10n+4}(1+q^{6n+5})}{(1-q^{6n+5})}$$

$$=\frac{(q^6;q^6)_\infty(q^2;q^2)_\infty}{(q,q^5;q^6)^2_\infty}. \tag{3.20}$$

Again, taking $\lambda=\mu=1, i=3, j=1$ and k=6 in (2.2) we find,

$$\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{(1-q^{6n+1})}=\sum_{n=0}^{\infty}\frac{q^{6n^2+4n}(1-q^{12n+4})}{(1-q^{6n+1})(1-q^{6n+3})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+8n+2}(1-q^{12n+8})}{(1-q^{6n+3})(1-q^{6n+5})}$$

$$=\frac{(q^6;q^6)_\infty(q^2;q^2)_\infty}{(q,q^5;q^6)_\infty(q^3;q^6)^2_\infty}. \tag{3.21}$$

Dividing (3.20) by (3.21) and using [1; corollary (6.2.7) p. 154] we get,

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^n}{(1-q^{6n+1})}}{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{(1-q^{6n+1})}}=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+2n}(1+q^{6n+1})}{(1-q^{6n+1})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+10n+4}(1+q^{6n+5})}{(1-q^{6n+5})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+4n}(1-q^{12n+4})}{(1-q^{6n+1})(1-q^{6n+3})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+8n+2}(1-q^{12n+8})}{(1-q^{6n+3})(1-q^{6n+5})}}$$

$$=\frac{(q^3;q^6)^2_\infty}{(q,q^5;q^6)_\infty}=1+\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^4+q^8}{1+\ldots}. \tag{3.22}$$

(x) Taking $k=8, \lambda=\mu=1, i=j=1$ in (2.2) we get,

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{(1-q^{8n+1})}=\sum_{n=0}^{\infty}\frac{q^{8n^2+2n}(1+q^{8n+1})}{(1-q^{8n+1})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+14n+6}(1+q^{8n+7})}{(1-q^{8n+7})}$$

$$=\frac{(q^8;q^8)^2_\infty(q^2;q^4)_\infty}{(q,q^7;q^8)^2_\infty}. \tag{3.23}$$

Again, taking $k=8, \lambda=\mu=1, i=j=3$ in (2.2) we have

$$\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{(1-q^{8n+3})}=\sum_{n=0}^{\infty}\frac{q^{8n^2+6n}(1+q^{8n+3})}{(1-q^{8n+3})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+10n+2}(1+q^{8n+5})}{(1-q^{8n+5})}$$

$$=\frac{(q^8;q^8)^2_\infty(q^2;q^4)_\infty}{(q^3,q^5;q^8)^2_\infty}. \tag{3.24}$$

Dividing (3.24) by (3.23) and using [1; (6.2.38) p. 154] we find,

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{(1-q^{8n+3})}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{n}}{(1-q^{8n+1})}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{8n^2+6n}(1+q^{8n+3})}{(1-q^{8n+3})} - \sum_{n=0}^{\infty} \frac{q^{8n^2+10n+2}(1+q^{8n+5})}{(1-q^{8n+5})}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{8n^2+2n}(1+q^{8n+1})}{(1-q^{8n+1})} - \sum_{n=0}^{\infty} \frac{q^{8n^2+14n+6}(1+q^{8n+7})}{(1-q^{8n+7})}}$$

$$= \left\{ \frac{(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty} \right\}^2 = \left\{ \frac{1}{1+}\, \frac{q+q^2}{1+}\, \frac{q^4}{1+}\, \frac{q^3+q^6}{1+}\, \frac{q^8}{1+}\, \frac{q^5+q^{10}}{1+}\, \frac{q^{12}}{1+\ldots} \right\}^2. \tag{3.25}$$

Similar other results can also be scored.

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